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Associative and Non-Associative Automorphisms of Newtonian Space-Time

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Abstract

In this article, an algebraic study of the transformations

$$(\mathbf{x}, t) \mapsto \left(R\cdot\mathbf{x} + \sum_{i=0}^{m} \mathbf{A}_i t^i/i!, t + t'\right)$$

of Newtonian space-time onto itself is presented. The study is carried out within the framework of Eilenberg and Maclane's co-homology theory of group prolongations, a generalisation of the theory of group extensions (Eilenberg & Maclane, 1947a–d). The 'loops' or non-associative groups involving 'm' from 0 to 5 are placed in classes of decreasing associativity described in the text. We also discuss the physical applications of the cup-product of co-chains and the vector bundle structure of Newtonian space-time.

1. *Newtonian Space-Time*

As a basis for our discussion of the automorphisms of Newtonian space-time, we choose the axioms and definitions formulated by Noll (1967). His definition of 'world automorphism' is extended from the original to one where, more generally, we can discuss the action of a set of non-associative automorphisms in the space-time.

After stating and developing Noll's axioms, we proceed to the point of obtaining a multiplication table for the generalised automorphisms. At this stage, we discuss the mathematical apparatus used to analyse the multiplication tables. Initially, the definitions and their immediate consequences are presented without the elaboration of proofs. We do, however, provide proofs for the most important theorems which are later used to classify the world automorphisms. Then, having developed the theory to a sufficient degree, we place the 'loops' of automorphisms in the relevant categories of group prolongation.

Axiomatic Newtonian Relativity

Let W be the set of events constituting the world. The first axiom concerns the measurement of time.

Axiom 1

There exists a function τ_0: $W \times W \to \mathbf{R}$ called the time-lapse function. For two events $x_1, x_2 \in W$, $\tau_0(x_1, x_2)$ is their relative time lapse. Experiment requires that τ_0 must satisfy the following three conditions:

(1) $\tau_0(x_1, x_2) = -\tau_0(x_2, x_1) \ \forall \ x_1, x_2 \in W$

(2) $\tau_0(x_1, x_2) + \tau_0(x_2, x_3) = \tau_0(x_1, x_3) \ \forall \ x_1, x_2, x_3 \in W$

(3) $\forall \ (x, t) \in W \times \mathbf{R}, \exists y \in W \ \ni \tau_0(x, y) = t$

One can assign relations in $W \times W$ via the function τ_0. Firstly,

$$F \equiv \{(x_1, x_2) \in W \times W \ \ni \tau_0(x_1, \mathbf{x}_2) \geqslant 0\}$$

F is called the future relation. The relation $P \equiv F^T$ is called the past relation. Forming the subsets

$$P(x) \equiv \{y \in W \ \ni (y, x) \in P\}, \qquad F(x) = \{y \in W \ \ni (y, x) \in F\}$$

Condition (1) means that $x \in F(y)$ iff $y \in P(x)$. Condition (2) means that F is a transitive relation, $F \circ F \subset F$. One can also define an equivalence relation $S \equiv F \cap P$ called 'simultenaity', where if $(x_1, x_2) \in S$, x_1 and x_2 are said to be simultaneous. Given $x \in W$, the subset

$$F(x) \cap P(x) = S(x)$$

is called the instant of x, the set of events in W 'simultaneous' with x. The world set W is thus partitioned into disjoint instants. Note that F is not a partial-order relation, since S is a proper subset. In special relativity, F is replaced with the Zeeman causal order (Zeeman, 1964) which is a partial order in W.

The second axiom of Newtonian relativity is concerned with the instants. It will be more convenient for us to work in terms of a function τ, called the relative time, instead of the time-lapse function τ_0. We select an event $z \in W$ as a base point. Given τ_0, a function τ_z: $W \to \mathbf{R}$ is defined called the 'time relative to the event z'. The function

$$\tau_z \colon x \mapsto \tau_0(x, z) \ \forall \ x \in W$$

is called the time of x relative to z. Given two events $x, y \in W$, the time lapse $\tau_0(x, y)$ in terms of the relative times is

$$\tau_0(x, y) = \tau_z(x) - \tau_z(y) \tag{1.1}$$

from condition (ii) on τ_0. Condition (3) in Axiom 1 means that $\forall z \in W$, τ_z is a surjective function. Thus W is a fibre bundle over $\mathbf{R}$. The cross-sections of W are trajectories in space-time. Axiom 2 imposes structure on the fibres or instants.

Axiom 2

Each fibre is a real, Euclidian linear space of three dimensions.

Thus Axioms 1 and 2 impose a vector bundle over $\mathbf{R}$ structure for W, and in addition; each fibre is an Euclidian topological space. The real line, '$\mathbf{R}$', is interpreted as the time axis.

Axiom 2 also implies the existence of a surjective function

$$\phi_z: W \to \mathbf{R}^3$$

where $\phi_z(x)$ is the position of the event $x \in W$ with respect to the event z, $\phi_z(z) = 0$. A bijective function γ_z: $W \to \mathbf{R}^3 \times \mathbf{R}$ can be defined in terms of τ_z and ϕ_z by

$$\gamma_z: x \mapsto (\phi_z(x), \tau_z(x))$$

In the following, we will drop the subscripts 'z'; working in terms of a fixed origin or present.

Having imposed the structure of a vector bundle of real, three-dimensional Euclidian spaces on W, we are in a position to define the so-called world automorphisms of W.

Axiom 3

There exists a set $A(W)$ and a mapping $\wedge: A(W) \times W \to W$, $\wedge: (g,x) \mapsto g \cdot x \; \forall \; (g,x) \in A(W) \times W$ such that conditions (i), (ii) and (iii) are true.

(i) $\tau_0 \circ (g \times g) = \tau_0 \; \forall \; g \in A(W)$
where $g \times g$: $(x_1, x_2) \mapsto (g \cdot x_1, g \cdot x_2) \; \forall \; x_1, x_2 \in W$

(ii) $A(W)$ is a loop and
$g_1 \cdot (g_2 \cdot x) = (g_1 \circ g_1) \cdot x \; \forall \; g_1, g_2 \in A(W), x \in W$

(iii) $e \cdot x = x \; \forall \; x \in W$, '$e$' being the idempotent of $A(W)$

We shall impose another axiom on the system shortly, but for the present we shall consider the consequences of Axiom 3.

Now there is, implied above, a loop homomorphism from L into the group Sym(W) of permutations of W, such that

$$\Phi(g)(x) = g \cdot x \; \forall \; g \in A(W), \qquad x \in W$$

Via the isomorphism γ: $W \cong \mathbf{R}^3 \times \mathbf{R}$, there is an action of $A(W)$ in $\mathbf{R}^3 \times \mathbf{R}$ and a homomorphism Φ^* from $A(W)$ to Sym$(\mathbf{R}^3 \times \mathbf{R})$.

The action of $A(W)$ on $\mathbf{R}^3 \times \mathbf{R}$ is defined by

$$g^* = \gamma \circ g \circ \gamma^{-1}$$

i.e.

$$g^* \gamma(x) = \gamma(g \cdot x)$$

or

$$g^*(\phi(x), \tau(x)) = (\phi(g \cdot x), \tau(g \cdot x))$$

The homomorphism Φ^* from $A(W)$ to $\mathrm{Sym}(\mathbf{R}^3 \times \mathbf{R})$ is just

$$\Phi^* = \Phi \circ I$$

Where $I: f \mapsto \gamma \circ f \circ \gamma^{-1}\ \forall f \in \mathrm{Sym}(W)$.

Now via axiom 3(i) we must have

$$\tau_0(g \cdot x_1, g \cdot x_2) = \tau_0(x_1, x_2)\ \forall\ g \in A(W);\ x_1, x_2 \in W$$

That is, from equation (1.1), we must have

$$\tau(g \cdot x_1) - \tau(g \cdot x_2) = \tau(x_1) - \tau(x_2)$$

or

$$\tau(g \cdot x_1) - \tau(x_1) = \tau(g \cdot x_2) - \tau(x_2)$$

We surmise that $\tau(g \cdot x_1) - \tau(x_1)$ is independent of x_1, i.e. we can define a function 'σ' from $A(W)$ onto $\mathbf{R}$ by

$$\tau(g \cdot x) = \tau(x) + \sigma(g)$$

By axiom 3(ii)

$$\sigma(g_1 \circ g_2) = \sigma(g_1) + \sigma(g_2) \tag{1.2}$$

Which means that $\sigma \in \mathrm{Hom}(A(W), \mathbf{R}^+)$, where $\mathbf{R}^+$ is the additive group of the reals.

We now formulate Axiom 4.

Axiom 4

For each $g \in \mathrm{Ker}(\sigma) \leqslant A(W)$, $\sigma(g)$ is an isometry of Euclidian space. That is, if x_1 and x_2 are two sections from R to W, then

$$\|x_1(t) - x_2(t)\| = \|\Phi(g)(x_1(t)) - \Phi(g)(x_2(t))\ \forall\ t \in \mathbf{R},\ g \in \mathrm{Ker}(\sigma)$$

It is clear then, that $\Phi(g)(x(t))$ is of the form $\Phi(g)(x(t)) = R(g)(x(t)) + f(g)(t)$, where R is a function from $A(W)$ onto $0(3, \mathbf{R})$ and $f(g)$ is a section from $\mathbf{R}$ to W. We can rewrite the above as

$$\Phi(g)(x(t)) = (R(g) \cdot x + f(g))(t)$$

or

$$g \cdot x(t) = R(g) \cdot x + f(g))(t)\ \forall\ g \in \mathrm{Ker}(\sigma) \tag{1.3}$$

From Axiom 3(ii) we must then have

$$g_1 \cdot g_2 x(t) = (R(g_1) \cdot R(g_2) x + R(g_1)(f)(g_2) + f(g_1))(t)$$

That is, $R \in \mathrm{Hom}(A(W), 0(3, \mathbf{R}))$ and

$$f \in Z^1_{can\,O\,\mathbf{R}}(A(W), \mathrm{Sec}(\mathbf{R}, \mathbf{R}^3))$$

Each element of $A(W)$ can be written

$$g = j(t) \circ i(R; f)$$

where i embeds the group extension $\mathrm{Sec}(\mathbf{R}, \mathbf{R}^3)\boxed{\times}_{can} O(3, \mathbf{R})$ into $A(W)$, which we identify with $\mathrm{Ker}(\sigma)$. 'j' is a section from $\mathbf{R}^+$ to $A(W)$ which we choose to be monomorphic

$$j(t_1) \circ j(t_2) = j(t_1 + t_2) \;\forall\; t_1, t_2 \in \mathbf{R}^+$$

We have

$$\tau(j(t)\cdot x) = \tau(x) + \sigma(j(t)) = \tau(x) + t$$

The action of the element $j(t) \circ i(R,f)$ on an event $x \in W$ is given by

$$\begin{aligned} j(t')^*(i(R,f)^*((\mathbf{x}, t)) &= j(t')^* \cdot (R\cdot\mathbf{x} + f(t), t) \\ &= (Rx + f(t), t + t') \end{aligned} \tag{1.4}$$

Thus applying equation (1.4) twice, we obtain

$$\begin{aligned} &[(j(t_1) \circ i(R_1,f_1)) \circ (j(t_2) \circ i(R_2,f_2))]^*(\mathbf{x}, t) \\ &\quad = [j(t_1) \circ i(R_1,f_1)]^* \circ [j(t_2) \circ i(R_2,f_2)]^*(\mathbf{x}, t) \\ &\quad = (R_1\cdot R_2\mathbf{x} + R_1\cdot f_2(t) + f_1(t + t_2), t + t_1 + t_2) \end{aligned} \tag{1.5}$$

Equation (1.5) thus establishes a multiplication table in $A(W)$. Its structure is the primary interest of this paper. We cannot as yet use the table to base a detailed analysis of the structure of $A(W)$, because we cannot 'add' the terms appearing as translations. To facilitate our discussion, we define a function

$$\xi: \mathrm{Sec}(\mathbf{R}, W) \times \mathbf{R} \to \mathrm{Sec}(\mathbf{R}, W)$$

by

$$\xi: (f, t): t' \mapsto f(t' + t) - f(t) \;\forall f \in \mathrm{Sec}(\mathbf{R}, W), t, t' \in \mathbf{R}$$

Equation (1.5) can then be re-written in the following more amenable form

$$\begin{aligned} (j(t_1) \circ i(R_1,f_1)) \circ (j(t_2) \circ i(R_2,f_2)) &= (t_1, (R_1,f_1))(t_2, (R_2,f_2)) \\ &= (t_1 + t_2, (R_1\cdot R_2, f_1 + R_1 f_2 + \xi(f_1, t_2)) \end{aligned} \tag{1.6}$$

Equation (1.6) will be used to determine the multiplication tables on loops $A(W)m \leqslant A(W) m \in \mathbf{Z}^+$. These sub-loops of $A(W)$ have the property that the 'translations' (viewed in $\mathrm{Sym}(\mathbf{R}^3 \times \mathbf{R})$) are polynomials in '$t$' of degree '$m$'.

Let us consider then polynomials of the form

$$f = \sum_{i=0}^{m} \mathbf{A}_i X^i$$

in the polynomial ring $\mathscr{P}(\mathbf{R}^3)$. We shall choose a substitution homomorphism of the form

$$\rho(t): f \mapsto \sum_{i=0}^{m} \mathbf{A}_i t^i / i!$$

We shall only in fact be interested in the abelian subgroups $\langle A_m \rangle m \in \mathbf{Z}^+$ of the abelian group $\mathscr{P}(\mathbf{R}^3)$

$$A_m \equiv \{f \in \mathscr{P}(\mathbf{R}^3) \;\}\!\!- \partial(f) \leqslant m \leqslant \infty\}$$

[$\partial(f)$ being the degree of f]. These abelian groups are clearly isomorphic to direct sums of isomorphic copies of $\mathbf{R}^3$

$$I_m\colon A_m \cong \bigoplus_{i=0}^{m} \mathbf{R}_i^{\,3}$$

via

$$I_m\colon \sum_{i=0}^{m} A_i X^i \mapsto \langle A_i \rangle$$

Physically, we interpret the action of these groups on W as follows. A_0 is the group of pure translations

$$(\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{A}_0, t)$$

Similarly, $\mathbf{R}_1^{\,3}$ consists of the group of pure velocity boosts, or Galilei boosts

$$(\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{A}_1 t, t)$$

whilst A_2 receives the interpretation of a Galilei boost with non-constant velocity

$$(\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{A}_0 + \mathbf{A}_1 t + \tfrac{1}{2}\mathbf{A}_2 t^2, t)$$

We shall call transformations in $\mathbf{R}_0^{\,3}$ translations; transformations in $\mathbf{R}_1^{\,3}$, velocity boosts, and transformations in $\mathbf{R}_2^{\,3}$ acceleration boosts.

Recall now equation (1.6) and the function $\xi(f_1, f_2)$. Limiting ourselves to the above polynomial sections (of a given degree m) we can find the form of ξ. Now

$$\xi(f_1, t_2)(t) = f_1(t) - f_1(t + t_2) = \sum_{i=0}^{m} \mathbf{A}_i^{\,1}(t^i - (t + t_2)^i)/i!$$

Expanding out powers of 't' we obtain

$$\xi(f_1, t_2)(t) = \sum_{i=0}^{m} \left(\sum_{j=0}^{m} t^{i-j}/(i-j)! \right) t^i \tag{1.7}$$

Thus, if we describe multiplication in $A(W)_m$ in terms of '$(m+3)$'-tuples

$$((\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m), (R, t)) \in \left(\bigoplus_{i=1}^{m} \mathbf{R}^3 \right) \times (0(3, \mathbf{R}) \times \mathbf{R}^+)$$

we obtain the multiplication table

$$((\mathbf{A}_0^{\,1}, \mathbf{A}_1^{\,1}, \ldots, \mathbf{A}_m^{\,1}), (R_1, t_1)) \circ ((\mathbf{A}_0^{\,2}, A_1^{\,2}, \ldots, \mathbf{A}_m^{\,2}), (R_2, t_2))$$

$$= \Bigg((R_1 \mathbf{A}_0^{\,2} + \sum_{j=0}^{m} A_j^{\,1} t_2^{\,j}/j!,\ R_1 \mathbf{A}_1^{\,2} + \sum_{j=m}^{m} \mathbf{A}_j^{\,1} t_2^{\,j-1}/(j-1)!, \ldots,$$

$$R_1 \mathbf{A}_k^{\,2} + \sum_{j=k}^{m} \mathbf{A}_j^{\,1} t_2^{\,j-k}/(j-k)!, \ldots, R_1 A_m^{\,2} + A_m^{\,1}, (R_1 R_2, t_1 + t_2) \Bigg)$$

Using the above multiplication table we shall investigate the structure of the subloops $A_m(W)$ of $A(W)$ for $m = 0$ to 5, in Section 3. Section 2 below is concerned with developing the necessary mathematical apparatus.

2. *Cohomology Theory of Group Prolongations* 3, 4, 5, 6, 7 *and* 8

The cohomology theory of group prolongations has been developed as a generalisation of the cohomology theory of group extensions. Only one paper has been published on this subject (Eilenberg & Maclane, 1947a), and papers on loops are rather few and far between. Hence here we derive and prove some of the most important theorems necessary for our programme. Central to the theory is the idea of not-necessarily exact sequences, and non-zero and zero sequences. Consider a sequence $C = \langle C^n, \delta^n \rangle n \in \mathbf{Z}^+$ of pairs, where, $\forall\, n \in \mathbf{Z}^+$, C^n is an additive abelian group and

$$\delta^n \in \mathrm{Hom}\,(C^n, C^{n+1})$$

One calls such a sequence a complex. If

$$\delta^n \circ \delta^{n-1} = O \;\forall\; n \in \mathbf{Z}^+$$

the sequence is called a 'zero sequence' or semi-exact sequence. Here we must have

$$\mathrm{Im}\,(\delta^{n-1}) < \mathrm{Ker}\,(\delta^n).$$

An exact sequence is a semi-exact sequence, with

$$\mathrm{Im}\,(\delta^{n-1}) = \mathrm{Ker}\,(\delta^n)$$

The group $C^n \in C$ is called the group of n-dimensional co-chains of the complex C. $\mathrm{Im}(\delta^{n-1}) \equiv B^n$ is called the group of n-dimensional co-boundaries, and $\mathrm{Ker}(\delta^n)$ the group of n-dimensional co-cycles of the complex. If C is a zero sequence, then $B^n < Z^n \;\forall\; n \in \mathbf{Z}^+$. In this case, the quotient group $H^n = Z^n/B^n$ is called the n-dimensional co-homology group of C. Clearly, if C is an exact sequence, $H^n = O \cdot \forall\; n \in \mathbf{Z}^+$.

Now, let Q be a multiplicative group and K be an abelian group, noted additively. Also let there be a function

$$p \in C(Q, \mathrm{Aut}\,(K))$$

[where $C(A,B)$ is the set of all functions from A to B]. Form the additive group, under addition of functions; $C_p{}^n(Q,K)$, of functions from Q^n to K. One defines a sequence $\langle \delta^n \rangle\; n \in \mathbf{Z}^+$ of homomorphisms

$$\delta^n \in \mathrm{Hom}\,(C_p{}^n(Q,K), C_p^{n+1}(Q,K))$$

via:

$$\delta^n(f)(q_1, \ldots, q_{n+1}) = p(q_1) \cdot f(q_2, \ldots, q_{n+1}) + (-1)^{n+1} f(q_1, \ldots q_n) + \sum_{i=1}^{n} (-1)^i f(q_1, \ldots, q_i \cdot q_{i+1}, \ldots, q_{n+1})$$

One can then verify (Eilenberg & Maclane, 1947b) that the sequence

$$Cp(Q, K) = \langle C_p{}^n(Q, K), \delta^n \rangle \, n \in \mathbf{Z}^+$$

has the property that:

$$\delta^{n+1} \circ \delta^n(f)(q_1, \ldots, q_{n+2}) = p(q_1 q_2 q_2) \cdot f(q_2, \ldots, q_{n+2}) \\ - p(q_1) \circ p(q_2) \cdot f(q_2, \ldots, q_{n+2})$$

Thus $Cp(Q, K)$ is a zero sequence iff

$$p \in \mathrm{Hom}\,(Q, \mathrm{Aut}\,(K))$$

In general then $Cp(Q, K)$ is a non-zero sequence. In our applications we shall have to deal both with zero and non-zero sequences of the above type.

In the following discussion of loops and group prolongations we shall very often just list the definitions and their consequences. The proofs of the theorems and lemmas left unproven are, on the whole, elementary.

2.1. *Loop Theory*

(i) A loop 'L' is set S with a binary operation which is a function from $S \times S$ to S.

(ii) $\exists\, e \in L\; e \circ \alpha = \alpha \circ e \;\forall\, \alpha \in L$.

(iii) $\forall\, \alpha, \beta \in L$, the equations $\alpha \circ x = \beta$, $y \circ \beta = \alpha$ have unique solutions.

It follows that the idempotent $e \in L$ is unique as in the group case. Iff L is associative, then axiom (iii) implies the identity and uniqueness of left and right inverses. In the general case, right and left inverse exist but are not identical.

Definition 1. We define a function $A_3 : L^3 \to L$ called the associator via

$$\alpha_1 \circ (\alpha_2 \circ \alpha_3) = A_3(\alpha_1, \alpha_2, \alpha_3) \circ ((\alpha_1 \circ \alpha_2) \circ \alpha_3) \\ \forall\, (\alpha_1, \alpha_2, \alpha_3) \in L^3$$

Higher associators are defined in terms of A_3 via

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n+1}) = A_3(\alpha_1, \alpha_2, A_{2n+1}(\alpha_3, \ldots, \alpha_{2n+1})) \\ \forall\, \alpha_1, \ldots, \alpha_{2n+1} \in L$$

Some Relevant Loop Theory

Definition 2. A sub-loop $L' \leqslant L$ is a subset of L which is closed multiplicatively and whose multiplication satisfies axioms (i) to (iii).

Definition 3. A subgroup $G < L$ is an associative sub-loop of L.

Definition 4. A function $f : L_1$ from a loop L_1 to a loop L_2 is a loop homomorphism if

$$f(\alpha_1 \circ \alpha_2) = f(\alpha_1) \circ f(\alpha_2) \;\forall\; \alpha_1, \alpha_2 \in L_1$$

Definition 5. A normal sub-group $G \lhd L$ is a sub-group of L for which

$$\alpha \circ G = G \circ \alpha \;\forall\; \alpha \in L$$

Definition 6. An element $\alpha \in L$ is said to be either left, centre or right associative in L if either, $\forall\ \beta, \gamma \in L$,

$$A_3(\alpha, \beta, \gamma) = e$$

or

$$A_3(\beta, \alpha, \gamma) = e$$

or

$$A_3(\beta, \gamma, \alpha) = e$$

respectively. A subset $S \subset L$ is right, left or centre associative in L if all its elements are right, left or centre associative. We state, without proof, the following propositions and lemmas.

Proposition 1. A sub-loop $L' \leqslant L$ is a sub-group $L' < L$ if it is either left and centre or right and centre associative in L.

Proposition 2. If G is a sub-group, centre associative in L, then the relation

$$\alpha_2 \sim a_2 \text{ iff } \exists\, g \in G \ni \alpha_1 = \alpha_2 \circ g$$

is an equivalence relation.

Lemma 1. Let G be a normal left and centre associative sub-group of L. Then

$$(G \circ \alpha) \circ (G \circ \beta) = G \circ (\alpha \circ \beta)\ \forall\ \alpha, \beta \in L$$

Lemma 2. If G is a centre and left associative normal sub-group of L, then the surjective map $\pi: L \to L/G$, $\pi\alpha: \mapsto G \circ \alpha$ is a loop homomorphism.

Lemma 3. The loop L/G, where G is left and centre associative in L and a normal sub-group, is a group if $A_3(\alpha_1, \alpha_2, \alpha_3) \in G\ \forall\ \alpha_1, \alpha_2, \alpha_3 \in L$.

Definition 7. Let α, $\beta \in L$. The inner transformation $\beta \mapsto \alpha \cdot \beta$ is defined $\forall\ \alpha, \beta \in L$ by

$$\alpha \circ \beta = \alpha \cdot \beta \circ \alpha$$

Lemma 4. Let $G < L$. Then $G \lhd L$ iff $\alpha \cdot g \in G\ \forall\ \alpha \in L, g \in G$.

Lemma 5. Let G be a normal, left and centre associative sub-group in L. Then $\forall\ \alpha \in L$, $g \mapsto \alpha \cdot g$ is in $\mathrm{Aut}(G)$. Also,

$$\alpha \cdot (\beta \cdot g) = A(\alpha, \beta, g) \circ (\alpha \circ \beta) \cdot g\ \forall\ \alpha, \beta \in L, g \in G$$

Group Prolongations

Definition 8. Let Q be a multiplicative group, K a left Q module. A prolongation of Q by K is a pair (L, ϕ) where

(i) L is a loop and K is a sub-loop of L.
(ii) $\phi \in \mathrm{Hom}(L, Q)$ and is onto, $\mathrm{Ker}(\phi) = G > K$.
(iii) G is left and centre associative in L.
(iv) The associators lie in $\mathscr{C}(G)$ the centre of G:

$$A_3(\alpha_1, \alpha_2, \alpha_3) \in \mathscr{C}(G)\ \forall\ \alpha_1, \alpha_2, \alpha_3 \in L$$

(v) $k = p(\phi(\alpha))(k)\ \forall\ \alpha \in L,\ k \in K$. $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ being the action of Q in its module K.

The following short lemmas are immediate consequences of the definition.

Lemma 6. G is a normal sub-group of L.

Lemma 7. K is a sub-group of $\mathscr{C}(G)$, the centre of G.

Lemma 8. K is the left, right and centre associative in L.

Lemma 9. $\mathscr{C}(G)$, the centre of G, is normal sub-group of L.

We see that the lack of associativity in a group prolongation is limited by the fact that the sub-loops $\mathscr{C}(G)$ and G are associative and normal and the quotient loops $L/\mathscr{C}(G)$ and L/G are also groups. If $G = K$ and L is associative (L, ϕ) is clearly a group extension of Q by K.

Characteristic Co-chains

Let (L, ϕ) be a prolongation of Q by K. Then by Definition 8(iv), we can regard the associators A_{2n+1} as 'co-chains' in:

$$C^{2n+1}_{can}(L, \mathscr{C}(G)) \ \forall\ n \geqslant 1$$

We form the *non-zero* sequence $\langle C^{n}_{can}(L, \mathscr{C}(G)), \delta^n\rangle$, where $\mathscr{C}(G)$ is mapped into itself automorphically by L in the canonical manner

$$can(\alpha): g \mapsto \alpha \cdot g \ \forall\ \alpha \in L, \qquad g \in G$$

Note that from Lemma 6, can $\notin$ Hom$(L$, Aut$(\mathscr{C}(G))$. We shall show that $\delta(A_{2n+1}) = O \ \forall\ n \geqslant 1$, or

$$A_{2n+1} \in Z^{2n+1}_{can}(L, \mathscr{C}(G))$$

Theorem 1

Let (L, ϕ) be a prolongation of Q by G. Then the associators A_{2n+1} are 'co-cycles' of the non-zero sequence $\mathscr{C}_{can}(L, \mathscr{C}(G))$

$$A_{2n+1} \in Z^{2n+1}_{can}(L, \mathscr{C}(G))$$

Outline of Proof. Clearly, $A_{2n+1} \in C^{2n+1}_{can}(L, \mathscr{C}(G))$ via Definition 8(iv). We shall show that $A_3 \in Z^3_{can}(L, \mathscr{C}(G))$, and proceed by induction. Firstly, we compute the product $\alpha_1 \circ (\alpha_2 \circ (\alpha_3 \circ \alpha_4))$ two ways. We have

$$\alpha_1 \circ (\alpha_2 \circ (\alpha_3 \circ \alpha_4)) = i(A_3(\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4)) \circ ((\alpha_1 \circ \alpha_2) \circ (\alpha_3 \circ \alpha_4))$$

where i injects $\mathscr{C}(G)$ into L. Thus

$$\begin{aligned}\alpha_1 \circ (\alpha_2 \circ (\alpha_3 \circ \alpha_4)) = i(&A_3(\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4) \\ &+ A_3(\alpha_1 \circ \alpha_2, \alpha_3, \alpha_4)) \circ ((\alpha_1 \circ \alpha_2) \circ {}_3) \circ {}_4)\end{aligned}$$

Again,

$$\begin{aligned}\alpha_1 \circ (\alpha_2 \circ (\alpha_3 \circ \alpha_4)) &= \alpha_1 \circ i(A_3(\alpha_2, \alpha_3, \alpha_4)) \circ ((\alpha_2 \circ \alpha_3) \circ \alpha_4) \\ &= i(\alpha_1 \cdot A_3(\alpha_2, \alpha_3, \alpha_4)) \circ (\alpha_1 \circ ((\alpha_2 \circ \alpha_3) \circ \alpha_4)) \\ &= i(\alpha_1 \cdot A_3(\alpha_2, \alpha_3, \alpha_4) + A_3(\alpha_1, \alpha_2 \circ \alpha_3, \alpha_4)) \\ &\qquad \circ (\alpha_1 \circ (\alpha_2 \circ \alpha_3)) \circ \alpha_4).\end{aligned}$$

$$\begin{aligned}i(\alpha_1 \cdot A_3(\alpha_2, \alpha_3, \alpha_4) &+ A_3(\alpha_1, \alpha_2 \circ \alpha_3, \alpha_4) \\ &+ A_3(\alpha_1, \alpha_2, \alpha_3)) \circ ((\alpha_1 \circ \alpha_2) \circ \alpha_3) \circ \alpha_4.\end{aligned}$$

Hence we must have

$$\alpha_1 \cdot A_3(\alpha_2, \alpha_3, \alpha_4) + A_3(\alpha_1, \alpha_2 \circ \alpha_3, \alpha_4) + A(\alpha_1, \alpha_2, \alpha_3) - A_3(\alpha_1 \circ \alpha_2, \alpha_3, \alpha_4) - A_3(\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4) = 0$$

Which is

$$\tilde{\delta}(A_3)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0, \; A_3 \in Z^3_{can}(L, \mathscr{C}(G))$$

One now proceeds by induction. However, we omit the remainder of this proof, which can be found in the work of Eilenberg & Maclane (1947a).

Now, let (L, ϕ) be a prolongation of Q by K. Choose a section $j: Q \to L$, $\phi \circ j = \mathbf{1}_Q$. Then we must have

$$j(q) \circ j(q_2) = g_j{}^2(q_1, q_2) \circ j(q_1 \circ q_2)$$

Where $g_j{}^2(q_1, q_2) \in G$, with $g_j{}^2(q_1, e) = g_j{}^2(e, q_1) = e$.

Definition 11. Given the canonical function $g_j{}^2$ from Q^2 to L corresponding to the section 'j' from Q to L, higher 'characteristic co-chains' of the non-zero sequence $\mathscr{C}_{can}(Q, \mathscr{C}(G))$ are defined by

$$g_j^{2n+1}(q_1, \ldots, q_{2n+1}) \equiv A_{2n+1}(j(q_1), \ldots, j(q_{2n+1}))$$

$$g_j^{2n+2}(q_1, \ldots, q_{2n+2}) \equiv A_{2n+1}(j(q_1), \ldots, g_j{}^2(q_{2n+1}, q_{2n+2}))$$

$\forall\, n \geqslant 1$. The canonical action of Q on $\mathscr{C}(G)$ is defined via

$$q : g \mapsto j(q) \cdot g \;\forall\, q \in Q, \; g \in \mathscr{C}(G)$$

Theorem 2

Considered as co-chains of the non-zero sequence $\mathscr{C}_{can}(Q, \mathscr{C}(G))$, the characteristic co-chains $g_j{}^n, n > 2$ are interrelated by

$$g_j^{n+1} = \tilde{\delta}(g_j{}^n) \;\forall\, n > 2$$

for a given section from Q to L. The expression can also be regarded as true for $n = 2$, when $g_j{}^2 \notin C^2_{can}(Q, \mathscr{C}(G))$ and $g_j{}^3 \in B^3_{can}(Q, \mathscr{C}(G))$.

Lemma 11. Given Definition 11 for the higher characteristic co-chains for $n > 2$

$$g_j{}^n \in \mathscr{C}^n_{can}(Q, \mathscr{C}(G))$$

If we make a new choice of section j' from Q to L, the co-chains $g^n_{j'}$ are related to the $g_j{}^n$ by

$$g_{j'}^{2n+1} = g_j^{2n+1} + \Phi^{2n+1}_{j,j'}$$

$$g_{j'}^{2n+2} = g_j^{2n+2} + \tilde{\delta}(\Phi^{2n+1}_{j,j'})$$

Where $\Phi^{2n+1}_{j,j'} \in \mathscr{C}^{2n+1}_{can}(Q, \mathscr{C}(G))$ is defined by

$$\Phi^{2n+1}_{j,j'} = A_{2n+1}(j(q_1), \ldots, j(q_{2n}) j'(q_{2n+1}) j(q_{2n+1})^{-1})$$

Types of Prolongations

Here; one restricts the non-associativity of the general prolongation of Q by K by requiring that certain higher associators vanish.

Definition 12. We define four basic types of prolongations

$$\Theta_n{}^L, \Theta_n{}^G, K_n{}^L, K_n{}^G$$

via

$$(L, \phi) \in \Theta_n{}^L \text{ iff } A_{2n+1}(\alpha_1, \ldots, \alpha_{2n+1}) = e$$
$$(L, \phi) \in \Theta_n{}^G \text{ iff } A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, g) = e$$
$$(L, \phi) \in K_n{}^L \text{ iff } A_{2n+1}(\alpha_1, \ldots, \alpha_{2n+1}) \in K$$
$$(L, \phi) \in K_n{}^G \text{ iff } A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, g) \in K$$

Where $\alpha_1, \ldots, \alpha_{2n+1} \in L$, $g \in G$.

From now on, we prove our theorems because of their importance in the following analysis in Section 3.

Lemma 12. The inclusions between the classes $\Theta_n{}^L$, $\Theta_n{}^G$, $K_n{}^L$ and $K_n{}^G$ are summarised by the non-exact diagram below.

$$\begin{tikzcd}
{} \arrow[r] & \Theta_n^L \arrow[r] \arrow[dr] & \Theta_n^G \arrow[r] \arrow[dr] & \Theta_{n+1}^L \arrow[r] \arrow[dr] & \Theta_{n+1}^G \arrow[r] & {} \\
{} \arrow[r] & K_{n-1}^G \arrow[r] \arrow[ur] & K_n^L \arrow[r] \arrow[ur] & K_n^G \arrow[r] \arrow[ur] & K_{n+1}^L \arrow[r] & {}
\end{tikzcd}$$

Proof. The inclusions $\Theta_n{}^L \subset \Theta_n{}^G$, $K_n{}^L \subset K_n{}^G$, $\Theta_n{}^L \subset K_n{}^L$ and $\Theta_n{}^G \subset K_n{}^G$ follow immediately from the definition. Now $\Theta_n \subset \Theta_{n+1}^L$, for, let $(L, \phi) \in \Theta_n{}^G$ then

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, g) = e$$

Hence

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, A_3(\alpha_{2n+1}, \alpha_{2n+2}, \alpha_{2n+3})) = e$$

i.e.

$$A_{2n+3}(\alpha_1, \ldots, \alpha_{2n+3}) = e$$

$(L, \phi) \in \Theta_{n+1}^L$, and $\Theta_n{}^G \subset \Theta_{n+1}^L$. Next let $(L, \phi) \in K_n{}^G$, then

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, g) \in K$$

therefore,

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n}, A_3(\alpha_{2n+1}, \alpha_{2n+2}, \alpha_{2n+3})) = A_{2n+3}(\alpha_1, \ldots, \alpha_{2n+3}) \in K$$

so $(L, \phi) \in K_{n+1}^L$, thus it follows that $K_n{}^G \subset K_{n+1}^L$. Lastly, $K_n{}^L \subset \Theta_{n+1}^L$ for, if $(L, \phi) \in K_n{}^L$, then

$$A_{2n+1}(\alpha_1, \ldots, \alpha_{2n+1}) \in K$$

But K is left, centre and right associative in L. So

$$A_3(\alpha_1, \alpha_2, A_{2n+1}(\alpha_3, \ldots, \alpha_{2n+3})) = 0$$

or

$$A_{2n+3}(\alpha_1, \ldots, \alpha_{2n+3}) = 0$$

so $(L, \phi) \in \Theta_{n+1}^L$. Hence $K_n{}^L \subset \Theta_{n+1}^L$.∎

Lemma 13. Let g_j^n be the characteristic co-chains of $\mathscr{C}^n_{can}(Q,\mathscr{C}(G))\, n > 2$ corresponding to the section $j: Q \to L$ in the prolongation (L,ϕ) of Q by K. Then

(i) $(L,\phi) \in \Theta_n{}^L \Rightarrow g_j^{2n+1} = 0$

(ii) $(L,\phi) \in \Theta_n{}^G \Rightarrow g_j^{2n+2} = 0$

(iii) $(L,\phi) \in K_n{}^L \Rightarrow g_j^{2n+1} \in \mathscr{C}_p^{2n+1}(Q,K)$

(iv) $(L,\phi) \in K_n{}^G \Rightarrow g_j^{2n+2} \in \mathscr{C}_p^{2n+2}(Q,K)$

Where the sequence $\langle (C_p{}^n(Q,K),\delta^n)\rangle n \in Z^+$ is a zero sequence, $p \in \mathrm{Hom}(Q, Aut(K))$.

Proof. (i) If $(L,\phi) \in \Theta_n{}^L$, then:

$$A_{2n+1}(q_1,\ldots,q_{2n+1}) = 0$$

thus, in particular,

$$A_{2n+1}(j(q_1),\ldots,j(q_{2n+1})) = 0$$

or

$$g_{2n+1}(q_1,\ldots,q_{2n+1}) = 0$$

(ii) If $(L,\phi) \in \Theta_n{}^G$, then

$$A_{2n+1}(q_1,\ldots,q_{2n},g) = 0, \qquad g \in G$$

in particular

$$A_{2n+1}(j(q_1),\ldots,j(q_{2n}),g_j{}^2(q_{2n+1},q_{2n+2})) = 0$$

or

$$g_j^{2n+2}(q_1,\ldots,q_{2n+2}) = 0$$

Let $(L,\phi) \in K_n{}^L$, then, by definition,

$$A_{2n+1}(\alpha_1,\ldots,\alpha_{2n+1}) \in K$$

or

$$g_j^{2n+1}(q_1,\ldots,q_{2n+1}) = A_{2n+1}(j(q_1),,\ldots,j(q_{2n+1})) \in K$$

So that $g_j^{2n+1} \in \mathscr{C}_p^{2n+1}(Q,K)$. Finally, (iv), if $(L,\phi) \in K_n{}^G$, then $A_{2n+1}(\alpha_1,\ldots,\alpha_{2n},g) \in K$, $g \in G$. Thus

$$A_{2n+1}(j(q_1),\ldots,j(q_{2n}),g_j{}^2(q_{2n+1},q_{2n+2})) = g_j^{2n+2}(q_1,\ldots,q_{2n+2}) \in K$$

so $g_j^{2n+2} \in \mathscr{C}_p^{2n+2}(Q,K)$.■

Theorem 3

If $n > 1$ and the prolongation $(L,\phi) \in \Theta_n{}^G \cap K_n{}^L$ then $g_j^{2n+i} \in Z_p^{2n+1}(Q,K)$, and $g_{j'}^{2n+1} = g_j^{2n+1}$ for all sections j' from Q to L. Also, if $(L,\phi) \in \Theta_n{}^L, g_j^{2n+1} = 0$ and if $(L,\phi) \in K_{n-1}^G$, then $g_j^{2n+2} \in B_p^{2n+1}(Q,K)$.

Proof. Let $n > 1$. If $(L,\phi) \in \Theta_n{}^G \cap K_n{}^L$ then, by Lemma 13,

$$g_j^{2n+2} = 0 = \delta(g_j^{2n+1}) \qquad \text{and} \qquad g_j^{2n+1} \in C_p^{2n+1}(Q,K)$$

Hence, $g_j^{2n+1} \in Z_p^{2n+1}(Q,K)$. If j' is another section of Q into L, then, via Lemma 11, we must have

$$g_{j'}^{2n+1} = g_j^{2n+1} + \Phi_{j,j'}^{2n+1}$$

where

$$\Phi_{j,j'}^{2n+1}(q_1,\ldots,q_{2n+1}) = A_{2n+1}(j(q_{2n}), j'(q_{2n}), j'(q_{2n+1})j(q_{2n+1})^{-1})$$

But

$$(L,\phi) \in \Theta_n{}^G \cap K_n{}^L \Rightarrow A_{2n+1}(g(q_1),\ldots,j(q_n), g) = 0 \ \forall\ g \in G$$

so

$$\Phi_{j,j'}^{2n+1}((q_1),\ldots,(q_n), q_{2n+1}) = 0$$

or $\Phi_{j,j'}^{2n+1} = 0$. Therefore, $g_{j'}^{2n+1} = g_j^{2n+1}\ \forall$ section j'.

Next, $(L,\phi) \in \Theta_n{}^L \Rightarrow g_j^{2n+1} = 0$ from Lemma 13. Also, if $(L,\phi) \in K_{n-1}^G$, then $g_j^{2n} \in C_p^{2n}(Q,K)$ by Lemma 13, so that $g_j^{2n+1} = \delta(g_j^{2n}) \in B_p^{2n+1}(Q,K)$.■

Theorem 4.

If $n > 1$ and $(L,\phi) \in \Theta_{n+1}^L \cap K_n{}^G$, then $g_j^{2n+2} \in Z_p^{2n+2}(Q,K)$, and if j' is an alternative choice of section of Q in L, then:

$$g_{j'}^{2n+2} = g_j^{2n+2} + \delta(\Phi_{j,j'}^{2n+1}),\ \Phi_{j,j'}^{2n+1} \in C_p^{2n+1}(Q,K)$$

$\Phi_{j,j'}^{2n+1}$ being defined as in Lemma 11. Also, if $(L,\phi) \in K_n{}^G$, then $g_j^{2n+2} = 0$, and if $(L,\phi) \in K_n{}^L$, then $g_j^{2n+2} \in B_p^{2n+2}(Q,K)$.

Proof. If $(L,\phi) \in \Theta_{n+1}^L \cap K_n{}^G$, then by Lemma 13 we must have

$$g_j^{2n+3} = 0 = \delta(g_j^{2n+2}) \qquad \text{and} \qquad g_j^{2n+2} \in C_p^{2n+2}(Q,K)$$

Thus, $g_j^{2n+1} \in Z_p^{2n+2}(Q,K)$. If j' is an alternative choice of section from Q to L, then, via Lemma 11, we must have

$$g_{j'}^{2n+2} = g_j^{2n+2} + \delta(\Phi_{j,j'}^{2n+1})$$

where, by Lemma 11, $\Phi_{j,j'}^{2n+1}$ is defined by

$$\Phi_{j,j'}^{2n+1}(q_1,\ldots,q_{2n+1}) = A_{2n+1}(j(q_1),\ldots,j(q_{2n}), j'(q_{2n+1})j(q_{2n+1})^{-1})$$

where $j'(q)(j(q))^{-1} G$. But we have $(L,\phi) \in \Theta_{n+1}^L \cap K_n{}^G$, so that

$$A_{2n+1}(j(q_1),\ldots,j(q_{2n}), j'(q_{2n+1})(j(q_{2n+1}))^{-1}) \in K$$

or $\Phi_{j,j'}^{2n+1} \in C_p^{2n+1}(Q,K)$. Now let $(L,\phi) \in \Theta_n{}^G$, by Lemma 13, $g_j^{2n+1} = 0$, and if $(L,\phi) \in K_n{}^L$, then $g_j^{2n} \in C_p^{2n}(Q,K)$, so that $g_j^{2n+1} = \delta(g^{2n}) \in B_p^{2n+1}(Q,K)$.■

In order to include the case when $G = K$, we define $K_0{}^G$ as the class of prolongations with $G = K$. Then

$$K_0{}^G \cap \Theta_1{}^L$$

is the class of associative prolongations of Q by K, the set of group extensions of Q by K. Now if

$$(L,\phi) \in K_0{}^G$$

then $g_j{}^2 \in \mathscr{C}_p{}^2(Q,K)$ and hence $g_j{}^3 = \delta(g_j{}^2) \in B_p{}^3(Q,K)$.

The higher canonical co-chains must vanish.

This completes our short review of the mathematical techniques to be applied in Section 3 below. Details of proofs omitted can be found in the work of Eilenberg & Maclane (1947a). A more elementary review can be found in the work of Eilenberg (1949), but this is much less detailed.

3. *The Loops* $\langle A(W)m\rangle 0 \leqslant m \leqslant 5$ *as Prolongations*

Here we compute 'by hand' the prolongations into which the $A(W)m$ can be case. The most physically relevant loop $A(W)_1$ will turn out to be a group, the Galilei group of inertial transformations.

Case 1*: The Loop* $A(W)_0$

The multiplication table of $A(W)_0$ is

$$((\mathbf{A}_0{}^1,(R_1,t_1))\circ((\mathbf{A}_0{}^2,(R_2,t_2))=((A_0{}^1+R_1\,\mathbf{A}_0{}^2,(R_1\cdot R_2,t_1+t_2))$$

This is a group whose structure of a group extension is trivial. That is

$$A(W)_0\cong E(3,\mathbf{R})_0\otimes\mathbf{R}^+$$

Here, $E(3,\mathbf{R})_0$ is the Euclidian group, a semi-direct product of $0(3,\mathbf{R})$ by $\mathbf{R}_0{}^3$ the translation group, via the canonical action of $0(3,\mathbf{R})$ on $\mathbf{R}^3$.

Case 2*: The Loop* $A(W)_1$ *(the Galilei Group)*

Multiplication in $A(W)_1$ takes the form

$$\begin{aligned}&((A_0{}^1,A_1{}^1),(R_1,t_1))\circ((A_0{}^2,A_1{}^2),(R_2,t_2))\\&\qquad=((A_0{}^1+R_1\cdot A_0{}^2+A_1\,t_2,A_{11}{}^1+R_1\cdot A_1{}^2),(R_1\,R_2,t_1+t_2))\end{aligned}$$

This loop has several interesting structures as a prolongation, some of them trivial. We shall derive the most interesting one which is relevant to the consideration of the higher loops, and list at the end the other structures. We exhibit, then, $A(W)_1$ as a prolongation of the group $E(3,\mathbf{R})_1\otimes\mathbf{R}^+$ by $\mathbf{R}^3$. Now the canonical epimorphism $\phi\in\mathrm{Hom}(A(W)_1,\,E(3,\mathbf{R})_1\otimes\mathbf{R}^+)$ is

$$\phi\colon((\mathbf{A}_0,\mathbf{A}_1),(R,t))\mapsto((\mathbf{A}_1,R),t)\tag{3.1}$$

Then $\mathrm{Ker}(\phi)\cong\mathbf{R}_0{}^3$ with an injection

$$\begin{aligned}&i\colon\mathbf{R}^3\mapsto A(W)_1\\&\qquad\rightarrowtail\\&i\colon\mathbf{A}_0\mapsto((\mathbf{A}_0,0),(e,0))\end{aligned}\tag{3.2}$$

Now there exists a homomorphism 'ρ'

$$\rho\in\mathrm{Hom}(E(3,\mathbf{R})_1,\,\mathrm{Aut}(\mathbf{R}^3))$$

with

$$\rho((\mathbf{A}_1,R),t)\colon\mathbf{A}_0\mapsto R\cdot\mathbf{A}_0\ \forall\ ((A_1,R),t)\in E(3,\mathbf{R})_1\otimes\mathbf{R}^+$$

with

$$j((A,R),t)\circ i(A_0)=i(\rho(A_1,R),t)(A_0))$$

where j is the section

$$j: ((\mathbf{A}_1, R), t) \mapsto ((0, A_1), (R, t))$$

to $A(W)_1$ from $E(3, \mathbf{R})_1 \otimes \mathbf{R}^+$. The canonical function

$$g_j^2 \in C^2(E(3, \mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}_0^3)$$

is given by

$$g_j^2(((A_1^1, R_1), t_1), ((A_1^2, R_2), t_2)) = A_2^1 t_2 \tag{3.3}$$

Now via Theorem 2, g_j^3 is δg_j^2, which is a co-boundary of $B_\rho^3(E(3, R)_1 \otimes \mathbf{R}^+, \mathbf{R}_0^3)$. But we have

$$\begin{aligned}
&\delta(g_j^2)(((A_1^1, R_1), t_1), ((A_1^2, R_2) t_2), ((A_1^3, R_3), t_3)) \\
&= \rho((A_1^1, R_1), t_1)(g_j^2((A_1^2, R_2), t_2), ((A_1^3, R_3), t_3)) \\
&\quad - g_j^2(((A_1^1 + R_1 A_1^2, R_1 R_2), t_1 + t_2), ((A_1^3, R_3), t_3)) \\
&\quad + g_j^2(((A_1^1, R_1), t_1), ((A_1^2 + R_2 A_1^3, R_2 R_3), t_2 + t_3)) \\
&\quad - g_j^2(((A_1^1, R_1), t_1), ((A_1^2, R_2), t_2) \\
&= R_1 \cdot A_1^2 t_3 - (A_1^1 + R_1 \cdot A_1^1) t_3 + A_1^1(t_2 + t_3) - A_1^1 t_2 = 0
\end{aligned}$$

Thus from Theorem 3, $(A(W)_1, \phi)$ is in $\Theta_1^{A(W)_1}$ and is associative, and hence a group. But since $\operatorname{Ker}(\phi) = K$, $A(W)_1$ is a group extension of $E(3, \mathbf{R}) \otimes \mathbf{R}^+$. $(A(W)_1, \phi) \in \Theta_1^{A(W)_1} \cap K_0^{\operatorname{Ker}(\phi)}$. Thus $A(W)_1$, which is the Galilei group, can be written as the group extension

$$R_0^3 \otimes g_j^2(E(3, \mathbf{R})_1 \otimes \mathbf{R}^+), \; g_j^2 \in Z_\rho^2(E(3, \mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}_0^3)$$

The other structures of $A(W)_1$ are that of a trivial group extension, being obtained by choosing epimorphisms ϕ_i

$$(A(W)_1, \phi_2) = (\mathbf{R}^3 \otimes \mathbf{R}^+) \boxtimes \rho_2 E(3, \mathbf{R})_1$$

$\rho_2 \in \operatorname{Hom}(E(3, \mathbf{R})_1, \operatorname{Aut}(\mathbf{R}^3 \otimes \mathbf{R}^+))$ being defined as

$$\rho_2((A_1, R)) : (A_0 t) \mapsto (R \cdot A_0 + A_1 t, t)$$

$$(A(W)_1, \phi_3) = (\mathbf{R}^3 \boxtimes \rho_3 E(3, \mathbf{R})_1) \boxtimes \mu \mathbf{R}^+$$

Where in $\operatorname{Ker}(\phi_3) = \mathbf{R}^3 \boxtimes \rho_3 E(3, \mathbf{R})_1$, the homomorphism ρ_3 is given by

$$\rho_3(A_1, R) : A_0 \mapsto R A_0$$

and the homomorphism $\mu \in \operatorname{Hom}(\mathbf{R}^+, \operatorname{Aut}(\operatorname{Ker}(\phi_3)))$ is

$$\mu(t) : (\mathbf{A}_0, R)) \to (\mathbf{A}_0 + \mathbf{A}_1 t, (\mathbf{A}_1, R))$$

$$(A(W)_1, \phi_4) = (\mathbf{R}^3 \otimes \mathbf{R}_1^3) \boxtimes \rho_4(0(3, \mathbf{R}) \otimes \mathbf{R}^+)$$

where $\rho_4 \in \operatorname{Hom}(0(3, \mathbf{R}) \otimes \mathbf{R}^+, \operatorname{Aut}(\mathbf{R}^3 \otimes \mathbf{R}_1^3))$ is defined by

$$\rho_4(R, t) : (\mathbf{x}, \mathbf{v}) \mapsto (R \cdot (\mathbf{x} - \mathbf{v}t), R \cdot \mathbf{v})$$

The group extension $(A(W)_1, \phi)$ is of most interest to us. Let us consider the co-cycle

$$g_j^2 \in Z_\rho^2(E(3,R)_1 \times R^+, R_0^3)$$

we can analyse g_j^2 into a so called 'cup-product' of certain 1-co-cycles. We discuss first the concept of cup-products.

'Cup-Products' of Co-chains

Let $\prod_1$, $\prod_2$ and $\prod$ be three G-modules, defined as such via $p_i \in \mathrm{Hom}(G, \mathrm{Aut} \prod_i))$ $i = 0, 1$ and 2. The modules $\prod_1$ and $\prod_2$ are said to be 'paired to $\prod$' if there is a function $P: \prod_1 \times \prod_2 \to \prod$ $P: (\pi_1, \pi_2) \mapsto \pi_1 \cup \pi_2 \; \forall \; (\pi_1, \pi_2) \in \prod_1 \times \prod_2$ such that

(i) $\pi_1 \cup (\pi_2 + \pi_2') = \pi_1 \cup \pi_2 + \pi_1 \cup \pi_{2'}$

(ii) $(\pi_1 + \pi_{1'}) \cup \pi_2 = \pi_1 \cup \pi_2{}^+ \, \pi_{1'} \cup \pi_{2'}$

(iii) $p(g)(\pi_1 \cup \pi_2) = p_1(g)(\pi_1) \cup p_2(g)(\pi_{2'})$

Let us now consider the zero-sequences

$$\mathscr{C}_{p1}(G, \textstyle\prod_1), \mathscr{C}_{p2}(G, \prod_2) \quad \text{and} \quad \mathscr{C}_p(G. \prod)$$

There exists a pairing P'

$$P': C_{p_1}^{m_1}(G, \textstyle\prod_1) \times C_{p_2}^{m_2}(G, \prod_2) \to C_p^{m_1+m_2}(G, \prod) P'(f_1, f_2) \mapsto f_1 \cup f_2$$

Where we define

$$f_1 \cup f_2(g_1, \ldots, g_{m_1+m_2}) - f_1(g, \ldots, g_{m_1}) \cup p_2(g_{m_1}, \ldots, g_{m_1})$$
$$(f(g_{m_1+1}\, g_{m_1+m_2})) \;\forall\; (g_1, \ldots, g_{m_1+m_2}) \in G^{m_1+m_2}$$

Moreover, one can show (Eilenberg & Maclane, 1947b) that if

$$f_1 \in Z_{p_1}^{m_1}(G, \textstyle\prod_1), f_2 \in Z_{p_2}^{m_2}(G, \prod_2)$$

then

$$f_1 \cup f_2 \in Z_p^{m_1+m_2}(G, \textstyle\prod)$$

whilst pairing of co-cycles and co-boundaries with co-boundaries produces co-boundaries.

Thus, given $f_1 \in Z_{p_1}^1(G, \prod_1), f_2 \in Z_{p_2}^1(G, \prod_2)$ there is a co-cycle

$$f_1 \cup f_2 \in Z_p^2(G, \textstyle\prod)$$

with

$$f_1 \cup f_2(g_1, g_2) = f_1(g_1) \cup p_2(g_1)(f(g_2))$$

Let us now consider the 2-co-cycle $g_j^2 \in Z^2(E(3, \mathbf{R}) \otimes \mathbf{R}^+, \mathbf{R}_0{}^3)$. Now the groups $\mathbf{R}_1{}^3$ and $\mathbf{R}^+$ can be paired to $\mathbf{R}_0{}^3$ via

$$\mathbf{A}_1 \cup t = \mathbf{A}_1 t \;\forall\; \mathbf{A}_1 \in \mathbf{R}_1{}^3, \qquad t \in \mathbf{R}^+$$

Consider the co-chains

$$f_1 \in C_{can_1}^1(E(3, \mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}_1{}^3)$$
$$f_1: ((\mathbf{A}_1, R), t) \mapsto \mathbf{A}_1$$

Clearly

$$f_1 \in Z^1_{can_1}(E(3,\mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}_1{}^3)$$

where $can_1((\mathbf{A}_1, R), t)\mathbf{A}_1 \to R\cdot\mathbf{A}_1$. Similarly, there is a 1-co-cycle

$f_2 \in Z^1_{can_2}(E(3,\mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}^+) = \mathrm{Hom}(E(3,\mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}^+) f_2 : ((\mathbf{A}_1, R), t) \mapsto t$

Forming the co-cycle $f_1 \cup f_2 \in Z_2{}^2(E(3,\mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}_0{}^3)$, we have

$$f_1 \cup f_2(((\mathbf{A}_1{}^1, R_1), t_1), ((\mathbf{A}_1{}^2, R_2), t_2) = \mathbf{A}_1{}^1 t_2$$

Thus we surmise that $g_j{}^2 = f_1 \cup f_2$. This notion of 'pairing' will be useful in the following work. We can pair $\mathbf{R}_i{}^3$ and $\mathbf{R}^+$ to $\mathbf{R}^3_{i-1}$ in exactly the same way as outlined above.

Case 3: The Loop $A(W)_2$

The multiplication table of $A(W)_2$ is

$$\begin{aligned}((\mathbf{A}_0{}^1, A_1{}^1, A_2{}^1), (R_1, t_1)) &\circ ((\mathbf{A}_0{}^2, A_1{}^2, A_2{}^2), (R_2, t_2)) \\ = ((\mathbf{A}_0{}^1 &+ R_1\mathbf{A}_0{}^2 + A_1{}^1 t_2 + \tfrac{1}{2}A_2 t_2{}^2, \mathbf{A}_1{}^1 + R_1\mathbf{A}_1{}^2 + \mathbf{A}_2{}^1 t_2, \\ &R_1\mathbf{A}_2{}^2 + \mathbf{A}_2{}^1), (R_1R_2, t_1 + t_2))\end{aligned}$$

We shall exhibit $A(W)_2$ in two different ways a group prolongation. Define firstly a homomorphism $\Phi_1 \in \mathrm{Hom}(A(W)_2, E(3,\mathbf{R})_2 \otimes \mathbf{R}^1)$ by

$$\Phi_1((\mathbf{A}_0, \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2), (R, t)) \mapsto ((A_2, R), t)$$

Then $\mathrm{Ker}(\Phi_1) \cong \mathbf{R}^3 \otimes \mathbf{R}_1{}^3$ with an injection

$$I_1 : \mathbf{R}^3 \otimes \mathbf{R}_1{}^3 \rightarrowtail A(W)_2$$

$$I_1 : (\mathbf{A}_0, A_1) \quad \mapsto ((\mathbf{A}_0, \mathbf{A}_1, 0), (e, 0))$$

The canonical section from $E(3,\mathbf{R})_2 \otimes R^+$ to $A(W)_2$ is

$$j: ((\mathbf{A}_2, R), t) \mapsto ((0, 0, \mathbf{A}_2), (R, t))$$

and if $\rho_2 \in \mathrm{Hom}(E(3,\mathbf{R})_2 \otimes \mathbf{R}^+, \mathrm{Aut}(\mathbf{R}_0{}^3 \otimes \mathbf{R}_1{}^3))$ is defined by

$$\rho_2((\mathbf{A}_2, R), t) : (\mathbf{A}_0, \mathbf{A}_1) \mapsto (R\cdot\mathbf{A}_0 + \mathbf{A}_1 t, R\mathbf{A}_1)$$

then

$$j((\mathbf{A}_2, R), t) \circ i(A_0, A_1) = i(\rho_2((A_2, R), t)(A_0, A_1))$$

The prolongation $(A(W)_2\Phi_2)$ is thus in the class $K_0^{\mathrm{Ker}(\Phi_1)}$. The canonical co-chain $g_j{}^2 \in C^2\rho_2(E(3,\mathbf{R})_1 \otimes \mathbf{R}^+, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$ is clearly

$$g_j{}^2(((A_2{}^1, R_1), t_1)((A_2{}^2, R_2), t_2) = (\tfrac{1}{2}A_1 t_2{}^2, A_1 t_2)$$

We must compute $g_j{}^3 = \delta(g_j{}^2) \in B^3(E(3,\mathbf{R})_2 \otimes \mathbf{R}^+, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$ to obtain a narrower classification of $(A(W)_2, \Phi_2)$. Now

$$\begin{aligned}\delta(g_j{}^2)&(((A_2{}^1, R_1), t_1), ((A_2{}^2, R_2), t_2), ((A_2{}^3, R_3), t_3)) \\ &= (\tfrac{1}{2}R_1 A_2{}^2 t_3{}^2 + A_2{}^2 t_3 t_1, R_1 A_2{}^2 t_3) - (\tfrac{1}{2}(A_2{}^1 + R_1 A_2{}^2) t_3{}^2, \\ &\qquad \times (A_2{}^1 + R_1 A_2{}^2) t_3) + (\tfrac{1}{2}A_2{}^1(t_2 + t_3)^2, A_2{}^1(t_2 + t_3)) \\ &\qquad - (\tfrac{1}{2}A_2{}^1 t_2{}^2, A_2{}^1 t_2) = (A_2{}^2 t_3 t_1 + A_2{}^1 t_2 t_3, 0)\end{aligned}$$

We see that the 3-co-boundary $\delta(g_j^2) = g_j^3$ does not vanish, and hence

$$(A(W)_2, \Phi_1) \in (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_0^{\mathrm{Ker}(\Phi)}$$
$$\notin (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_0^{\mathrm{Ker}(\Phi)} \cap \Theta_1^{\mathrm{Ker}(\Phi)}$$

Case 4: *The Loop* $A(W)_3$

Multiplication in $A(W)_3$ is defined by

$$\begin{aligned}((\mathbf{A}_0{}^1, A_1{}^1, A_2{}^1, A_3{}^1), (R_1, t_1)) \circ ((A_0{}^2, A_1{}^2, A_2{}^2, A_3{}^2), (R_2, t_2))\\ = ((A_0{}^1 + R_1 A_0{}^2 + \tfrac{1}{6}\mathbf{A}_3{}^1 t_2{}^3 + \tfrac{1}{2}A_2{}^1 t_2{}^2 + A_1{}^1 t_2,\\ \mathbf{A}_1{}^1 + R_1 A_1{}^2 + \tfrac{1}{2}\mathbf{A}_3{}^1 t_2{}^2 + A_2{}^1 t_2,\\ A_2{}^1 + R_1 A_2{}^2 + A_3{}^1 t_2, A_3{}^1 + R^1 A_3{}^2), (R_1 R_2, t_1 + t_2))\end{aligned}$$

$A(W)_3$ can be exhibited as a prolongation in the following way. Define $\Phi_1 \in \mathrm{Hom}(A(W)_3, A(W)_{1''})$

$$\Phi_1 : A(W)_3 \to A(W)_{1''}$$
$$\Phi_1 : ((A_0, A_1, A_2, A_3), (\mathbf{R}, t)) \mapsto (A_2, ((A_3, R), t))$$

Then $\mathrm{Ker}(\Phi_1)$ is isomorphic to $\mathbf{R}^3 \otimes \mathbf{R}_1{}^3$. The group $A(W)_{1''}$ has the analogous structure to the Galilei group:

$$\mathbf{R}_2{}^3 \otimes_{f_1'' \cup f_2''} (E(3, \mathbf{R})_3 \otimes \mathbf{R}^+)$$

Where the 2-co-cycle $f_1'' \cup f_2'' \in Z^2_{can}(E(3, \mathbf{R}) \otimes \mathbf{R}^+, \mathbf{R}_2{}^3)$ is defined by the pairing of the groups $\mathbf{R}_3{}^3$ and $\mathbf{R}^+$ to $\mathbf{R}_2{}^3$

$$\mathbf{A}_3 \cup t \equiv \mathbf{A}_3 t$$

and the canonical co-chains

$$f_{1''} \in Z^1_{can}(E(3, \mathbf{R})_3 \otimes \mathbf{R}^+, \mathbf{R}_3{}^3); \qquad f_{2''} \in Z^1_{can}(E(3, \mathbf{R})_3 \otimes \mathbf{R}^+, \mathbf{R}^+)$$

There is a homomorphism

$$\rho_4 \in \mathrm{Hom}(\mathbf{R}_2{}^3 \otimes_{f_1'' \cup f_2''} (E(3, \mathbf{R})_3 \otimes \mathbf{R}^+), \mathrm{Aut}(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3))$$

defined by

$$\rho_4((A_2, ((A_3, R), t) : (A_0, A_1) \mapsto (RA_0 + A_1 t, RA_1)$$

which in terms of the canonical section

$$j : A(W)_{1''} \to A(W)_3$$
$$j : (A_2, ((A_3, R), t) \mapsto ((0, 0, A_2, A_3), (R, t))$$

satisfies

$$j(A_2, ((A_3, R), t) \circ I_1((A_0, A_1)) = I_1(\rho_4(A_2, ((A_3, R), t))(A_0, A_1))$$

Thus $(A(W)_3, \Phi_1) \in (\mathbf{R}^3 \otimes \mathbf{R}_2{}^3)_0^{\mathrm{Ker}(\Phi_1)}$. The canonical co-chain of the prolongation

$$g_j{}^2 \in C^2_{\rho 4}(A(W)_{1''}, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)) =$$

is

$$\begin{aligned}g_j{}^2((A_2{}^1, ((A_3{}^1, R_1), t_1), (A_2{}^2, ((A_3{}^2, R_2), t_2))\\ = (\tfrac{1}{6}A_3{}^1 t_2{}^3 + \tfrac{1}{2}A_2{}^1 t_2{}^2, \tfrac{1}{2}A_3{}^1 t_2{}^2 + A_2{}^1 t_2)\end{aligned}$$

Then $g_j{}^3 = \delta(g_j{}^2)$ is a co-boundary of $B^3_{\rho_4}(A(W)_{1''}, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$ which does not vanish

$$g_j{}^3((A_2{}^1, ((A_3{}^1, R_1), t_1), ((A_2{}^2, ((A_3{}^2, R, t_2)), (A_2{}^3, R_3), t_3)))$$
$$= (\tfrac{1}{2}(A_3{}^1(t_2{}^2 t_3) + A_3{}^2(t_3{}^2 t_1)) + A_2{}^1 t_2 t_3 + A_2{}^2 t_3 t_1, 0)$$

Thus we surmise

$$(A(W)_3, \Phi_1) \in (\mathbf{R}^3 \otimes \mathbf{R}^3)_0^{\mathrm{Ker}(\Phi_1)}$$
$$(A(W)_3, \Phi_1) \in \Theta_1^{A(W)_3} \cap (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_0^{\mathrm{Ker}(\Phi_1)}$$

Case 5: The Loop $A(W)_4$

The multiplication in $A(W)_4$ is

$$((A_0{}^1, A_1{}^1, A_2{}^1, A_3{}^1, A_4{}^1), (R_1, t_1)) \circ ((A_0{}^2, A_1{}^2, A_2{}^2, A_3{}^2, A_4{}^2), (R_2, t_2))$$
$$= ((A_0{}^1 + R_1 A_0{}^2 + A_1{}^1 t_2 + \tfrac{1}{2} A_2{}^1 t_2{}^2 + \tfrac{1}{6} A_3{}^1 t_2{}^3 + \tfrac{1}{24} A_4 t_2{}^4,$$
$$A_1{}^1 + R_1 A_1{}^2 + A_2{}^1 t_2 + \tfrac{1}{2} A_3{}^1 t_2{}^2 + \tfrac{1}{6} A_4{}^1 t_2{}^3,$$
$$A_2{}^1 + R_1 A_2{}^2 + A_3{}^1 t_2 + A_3{}^1 t_2 + \tfrac{1}{2} A_4{}^1 t_2{}^2, A_3{}^1$$
$$+ R_1 A_3{}^2 + A_4{}^1 t_2, A_4{}^1 + R_1 A_4{}^2), (R_1 R_2, t_1 + t_2))$$

Firstly, we shall exhibit $A(W)_4$ as a prolongation of $A(W)_1'''$ (defined via the epimorphism Φ_1

$$\Phi_1 \in \mathrm{Hom}\,(A(W)_4, A(W)_1''')$$

by the group $\mathbf{R}^3 \otimes \mathbf{R}_1{}^3$. The epimorphism Φ_1 is

$$\Phi_1 : ((A_0, A_1, A_3, A_4), (R, t)) \mapsto (A_3, ((A_4, R), t))$$

and $\mathrm{Ker}(\Phi_1)$ is isomorphic to $(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3$, the latter being embedded in $A(W)_4$ via the monomorphism

$$I_1 : ((A_0, A_1), A_2) \mapsto (A_0, A_1, A_2, 0, 0), (e, 0))$$

The group $A(W)_1'''$ has a structure akin to the Galilei group

$$\mathbf{R}_3{}^3 \otimes (E(3, \mathbf{R})_4 \otimes \mathbf{R}^+) f_1''' \cup f_2'''$$

Where $f_1''' \cup f_2''' \in Z^2_{can}(E(3, \mathbf{R})_4 \otimes \mathbf{R}^+, \mathbf{R}_3{}^3)$ is defined by the pairing of $\mathbf{R}_4{}^3$ and $\mathbf{R}^+$ to $\mathbf{R}_3{}^3$

$$\mathbf{A}_4 \cup t = \mathbf{A}_4 t$$

and the canonical 1-co-cycles:

$$f_1''' \in Z^1_{can}(E(3, \mathbf{R})_4 \otimes \mathbf{R}^+, \mathbf{R}_4{}^3)$$
$$f_2''' \in Z^1_{can}(E(3, \mathbf{R})_4 \otimes \mathbf{R}^+, \mathbf{R}^+) = \mathrm{Hom}\,(E(3, \mathbf{R})_4 \otimes \mathbf{R}^+, \mathbf{R}^+)$$
$$f_1 \cup f_2(((A_4{}^1, R_1), t_1), ((A_4{}^2, R_2), t_2)) = A_4{}^1 \cup t_2$$

Now there exists a function

$$\rho_6 \in C^1(A(W)_1''', \mathrm{Aut}\,((\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3))$$
$$\rho_6 \notin \mathrm{Hom}\,(A(W)_1, \mathrm{Aut}\,((\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3))$$

defined by

$$\rho_6((A_4, R), t)((A_0, A_1), A_2) \mapsto (RA_0 + A_1 t + \tfrac{1}{2}A_2 t^2, RA_1 + A_2 t), RA_2)$$

such that if j is the section

$$j: A(W)_1''' \to A(W)_4$$
$$j: (A_3, ((A_4, R), t)) \mapsto ((0, 0, 0, A_3, A_4), (R, t))$$

then

$$j: (A_3, ((A_4, R), t) \cdot I_1((A_0, A_1), A_2) = I_1(\rho_6(A_3, ((A_4, R), t)((A_0, A_1), A_2)$$

if we define

$$\rho_6' \in C^{1(A(W))_1} \mathrm{Aut}((R^3 \otimes R_1{}^3) \otimes R_2{}^3))$$

by

$$I_1 \circ \rho_6'(A_3, ((A_4, R), t) = \rho_6(A_3, ((A_4, R), t) \circ I_1,$$

then

$$\rho_6 \in \mathrm{Hom}(A(W)_{1'}, \mathrm{Aut}(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3))$$

We must have $(A(W)_4, \Phi_j) \notin (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_0^{\mathrm{Ker}(\Phi_1)}$

The canonical 'co-chain' $g_j{}^2 \in C_6{}^2(A(W)_1, (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3)$ is given by

$$\begin{aligned} &g_j{}^2((A_3{}^1, ((A_4{}^1, R_1), t_1)), ((A_3{}^2, ((A_4{}^2, R_2), t_2))) \\ &\quad = ((\tfrac{1}{24}A_4{}^1 t_2{}^4 + \tfrac{1}{6}A_3{}^1 t_2{}^3, \tfrac{1}{6}A_4{}^1 t_2{}^3 + \tfrac{1}{2}A_3{}^1 t_2{}^2), \tfrac{1}{2}A_4{}^1 t_2{}^2 + A_3{}^1 t_2) \end{aligned}$$

We now compute the function $g_j{}^3 = \delta(g_j{}^2)$, we have

$$\begin{aligned} &g_j{}^3(A_3{}^1, ((A_4{}^1, \mathbf{R}_1), t_1)), ((A_3{}^2, ((R_4{}^2, R_2), t_2), (A_3{}^3, ((A_4{}^3, R_3), t_3)) \\ &\quad = \rho_6(A_3{}^1, ((A_4{}^1, R_1), t_1)(g_j{}^2((A_3{}^2, ((A_4{}^2, R_2), t_2)), ((A_3{}^3, ((A_4{}^3, R_3), t_3)) \\ &\qquad g_j{}^2((A_3{}^1 + R_1 A_3{}^2 + A_4{}^1 t_2, ((A_4{}^1 + R_1 A_4{}^2, R_1 R_2), t_1 + t_2)), \\ &\qquad ((A_3{}^3, ((A_4, R_3), t_3))) + g_j{}^2((A_3{}^1, ((A_4{}^1, R_1), t_1), \\ &\qquad (A_3{}^2 + R_2 A_3{}^3 + A_4{}^2 t_3, ((A_4{}^2 + R_2 A_4{}^3, R_1 R_2), t_2 + t_3)) \\ &\qquad\qquad - g_j{}^2((A_3{}^1, ((A_4{}^1, R_1), t_1)), (A_3{}^2, ((A_4{}^2, R_2), t_2))) \end{aligned}$$

After a long calculation we find that

$$\begin{aligned} &g_j{}^3((A_3{}^1, ((A_4{}^1, R_1), t_1)), (A_3{}^2, ((A_4{}^2, R_2), t_2)), (A_3{}^3, ((A_4{}^3, R_3), t_3))) \\ &\quad = ((\tfrac{1}{6}(A_4{}^1 t_2{}^3 t_3 + A_4{}^2 t_3{}^3 t_1) + \tfrac{1}{4}(A_4{}^1 t_2{}^2 t_3{}^2 + A_4{}^2 t_3{}^2 t_1{}^2) \\ &\qquad + \tfrac{1}{2}\mathbf{A}_3{}^1(t_2{}^2 t_3 + t_3{}^2 t_2) + \tfrac{1}{2}A_3{}^2(t_3{}^2 t_1 + t_1{}^2 t_3), \\ &\qquad\qquad \tfrac{1}{2}(A_4{}^1 t_2{}^2 t_3 + A_4{}^2 t_3{}^2 t_1) + A_3{}^1 t_2 t_3 + A_3{}^2 t_3 t_1), 0) \end{aligned}$$

Thus $g_j{}^3 \in C_{6'}{}^3(A(W)_{1''}, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$, so we must have

$$(A(W)_4, \Phi_1) \in (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_1^{A(W)_1}$$

For further information, we have to compute the 4-co-boundary

$$g_j{}^4 = \delta(g_j{}^3) \in B^4_{\rho 6'}(A(W)_{1'}, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$$

On computing the co-boundary, we find that $g_j{}^4 \neq 0$, so that

$$(A(W)_4, \Phi_1) \notin (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_1^{A(W)_4} \cap \Theta_1^{(^3\mathbf{R}\otimes\mathbf{R}_1{}^3)\otimes\mathbf{R}_2{}^3}$$

However, if we define the sub-loop $\tilde{A}(A)_4$ of $A(W)_4$ by the loop obtained when rotations are omitted, with multiplication table

$$((A_0{}^1, A_1{}^1, A_2{}^1, A_3{}^1, A_4{}^1), t_1) \circ ((A_0{}^2, A_1{}^2, A_2{}^2, A_3{}^2, A_4{}^2), t_2)$$
$$= ((A_0{}^1 + \tfrac{1}{24}A_4 t_2{}^4 + \tfrac{1}{6}A_3 t_2{}^3 + \tfrac{1}{2}A_2 t_2{}^2 + A_1 t_2 + A_0{}^2$$
$$\times A_1{}^1 + A_1{}^2 + \tfrac{1}{6}A_4 t_2{}^3 + \tfrac{1}{2}A_3 t_2{}^2 + A_2 t_2, A_2{}^1 + A_2{}^2$$
$$+ \tfrac{1}{2}A_4{}^1 t_2{}^2 + A_3 t_2, A_3{}^1 + A_3{}^2 + A_4{}^1 t_2, A_4{}^1 + A_4{}^2), t_1 + t_2)$$

and define a homomorphism $\Psi : \tilde{A}(W)_4 \to (\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+$ via

$$\Psi : ((A_0, A_1, A_2, A_3, A_4), t) \mapsto ((A_3, A_4), t)$$

Then $\mathrm{Ker}(\Psi)$ is isomorphic to $(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3$, the latter being embedded in $\tilde{A}(W)_4$ by

$$I : ((A_0, A_1), A_2) \mapsto ((A_0, A_1, A_2, 0, 0), (e, 0))$$

If

$$j : (\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^3 \to \tilde{A}(W)_4$$
$$j : ((A_3, A_4), t) \mapsto ((0, 0, 0, A_3, A_4), t)$$

is the canonical section from $(\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+$ to $\tilde{A}(W)_4$ and

$$\mu \in C^1((\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+, \mathrm{Aut}((\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3))$$
$$\mu \notin \mathrm{Hom}((R_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+, \mathrm{Aut}((\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3))$$

is

$$\mu((A_3, A_4), t) : ((A_0, A_1), A_2) \mapsto ((A_0 + \tfrac{1}{2}A_2 t^2 + A_1 t, A_1 + A_2 t), A_2)$$

Then, if J embeds $\mathbf{R}^3 \otimes \mathbf{R}_1{}^3$ into $(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3$

$$j((A_3, A_4), t) \cdot I((A_0, A_1), A_2) = I(\mu((A_3, A_4), t)((A_0, A_1), A_2))$$

and if μ' is defined by

$$\mu'((A_3, A_4), t) = \mu((A_3, A_4), t) \circ \Phi$$

and

$$\mu' \in \mathrm{Hom}((\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+, \mathrm{Aut}(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3))$$

Then $(\tilde{A}(W)_4, \Phi) \notin (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)_0^{\mathrm{Ker}(\Phi)}$.

The canonical 2-co-chain $g_j{}^2 \in C^2((\mathbf{R}_3{}^3 \otimes \mathbf{R}_4{}^3) \otimes \mathbf{R}^+, (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3)$ is

$$g_j{}^2((A_3{}^1, A_4{}^1), ((A_3{}^2, A_4{}^2), t^2))$$
$$= ((\tfrac{1}{24}A_4{}^1 t_2{}^4 + \tfrac{1}{6}A_3{}^1 t_2{}^3, \tfrac{1}{6}A_4{}^1 t_2{}^3 + \tfrac{1}{2}A_3{}^1 t_2{}^2), \tfrac{1}{2}A_4{}^1 t_2{}^2 + A_3{}^1 t_2)$$

The function $g_j{}^3 = \delta(g_j{}^2)$ is found to be

$$g_j{}^3(((A_3{}^1, A_4{}^1), t_1), ((A_3{}^2, A_4{}^2), t_2), ((A_3{}^3, A_4{}^3), t_3))$$
$$= ((\tfrac{1}{6}(A_4{}^1 t_2{}^3 t_3 + A_4{}^2 t_3{}^3 t_1) + \tfrac{1}{4}(A_4{}^1 t_2{}^2 t_3{}^2 + A_4{}^2 t_3{}^2 t_1{}^2)$$
$$+ \tfrac{1}{2}A_3{}^1(t_2{}^2 t_3 + t_3{}^2 t_2) + \tfrac{1}{2}A_3{}^2(t_3{}^2 t_1 + t_3 t_1{}^2),$$
$$\tfrac{1}{2}(A_4{}^1 t_2{}^2 t_3 + A_4{}^2 t_2{}^2 t_1) + A_3{}^1 t_2 t_3 + A_3{}^2 t_3 t_1), 0)$$

This is the same form as the co-chain we obtained for $(A(W)_4, \Phi)$. But, on computing the co-boundary $g_j{}^4 = \delta(g_j{}^3)$ we find that $g_j{}^4 = 0$. Hence,

$$g_j{}^3 \in Z^3_{\mu'}(\tilde{A}(W)_4, \mathbf{R}^3 \otimes \mathbf{R}_1{}^3)$$

with

$$(A(W)_4, \Phi) \in (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3)^{A(W)_4} \cap \Theta_1^{(\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes \mathbf{R}_2{}^3}$$

Case 6: The Loop $A(W)_5$

This is the last loop whose structure as a prolongation we will consider (the amount of tedious calculation increases as a very high power of 'm'!) The multiplication table for $A(W)_5$ is

$$\begin{aligned}
&((A_0{}^1, A_1{}^1, A_2{}^1, A_3{}^1, A_4{}^1, A_5{}^1), (R_1, t_1)) \circ ((A_0{}^2, A_1{}^2, A_2{}^2, A_3{}^2, A_4{}^2, A_5{}^2)(R_2, t) \\
&\quad = ((A_0{}^1 + R^1 A_0{}^2 + \tfrac{1}{120} A_5{}^1 t_2{}^5 + \tfrac{1}{24} A_4{}^1 t_2{}^4 + \tfrac{1}{6} A_3{}^1 t_2{}^3 + \tfrac{1}{2} A_2{}^1 t_2{}^2 + A_1 t_2, \\
&\qquad A_1{}^1 + R^1 A_1{}^2 + \tfrac{1}{24} A^1 t_2{}^4 + \tfrac{1}{6} A_4{}^1 t_2{}^3 + \tfrac{1}{2} A_3{}^1 t_2{}^2 + A_2{}^1 t_2, A_2{}^1 + R_1 A_2{}^2 \\
&\qquad + \tfrac{1}{6} A_5{}^1 t_2{}^3 + \tfrac{1}{2} A_4{}^1 t_2{}^2 + A_3{}^1 t_2, A_3{}^1 + R_1 A_3{}^2 + \tfrac{1}{2} A_5{}^1 t_2{}^2 + A_4{}^1 \\
&\qquad\qquad + R_1 A_4{}^2 + A_5{}^1 t_2, A_5{}^1 + R_1 A_5{}^2), (R_1 R_2, t_1 + t_2))
\end{aligned}$$

We exhibit $A(W)_5$ as a prolongation of $A(W)_1^{iv}$ by $\mathbf{R}^3 \otimes \mathbf{R}_1{}^3$. Define $\Phi_1 \in \mathrm{Hom}(A(W)_5, A(W)_1^{iv})$ by

$$\Phi_1 : ((A_0, A_1, A_2, A_3, A_4, A_5), (R, t)) \mapsto (A_4, ((A_5, R), t))$$

Then $\mathrm{Ker}(\Phi_1) \cong (\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes (\mathbf{R}_2{}^3 \otimes \mathbf{R}_3{}^3)$, which is embedded in $A(W)_5$ via the monomorphism

$$I_1 : ((A_0, A_1), (A_2, A_3)) \mapsto ((A_0, A_1, A_2, A_3) 0, 0), (e, 0))$$

As before, we can analyse the structure of the group $A(W)_1^{iv}$ into that of the group extension

$$\mathbf{R}_4{}^3 \otimes (E(3, \mathbf{R})_5 \otimes \mathbf{R}^+ f_1^{iv} \cup f_2^{iv}$$

where the co-cycle $f_1^{iv} \cup f_2^{iv} \in Z^2_{can}(E(3, \mathbf{R})_5 \otimes \mathbf{R}^+ \mathbf{R}_4{}^3)$ is defined by the pairing

$\mathbf{A}_5 \cup t \equiv \mathbf{A}_5 t$ of $\mathbf{R}_5{}^3$ and $\mathbf{R}^+$ to $\mathbf{R}_4{}^3$, and the canonical co-chains

$$f_1^{iv} \in Z^1_{can}(E(3, \mathbf{R})_5 \otimes \mathbf{R}^+, \mathbf{R}_5{}^3)$$
$$f_2^{iv} \in Z^1_{can}(E(3, \mathbf{R})_5 \otimes \mathbf{R}^+, \mathbf{R}^+)$$

The canonical section 'j' from $A(W)_1^{iv}$ to $A(W)_5$ is

$$j : (A_4, ((A_5, R), t)) \mapsto ((0, 0, 0, 0, \mathbf{A}_4, A_5), (R, t))$$

which, with the non-homomorphic function

$$\begin{aligned}
&\gamma \in C^1(A(W)_1^{iv}, \mathrm{Aut}((\mathbf{R}^3 \otimes \mathbf{R}_1{}^3) \otimes (\mathbf{R}_2{}^3 \otimes \mathbf{R}_3{}^3))) \\
&\qquad\qquad\qquad \gamma(A_4, ((A_5, R), t)((A_0, A_1), (A_2, A_3)) \\
&= ((RA_0 + \tfrac{1}{6} A_3 t^3 + \tfrac{1}{2} A_2 t^2 + A_1 t, RA_1 + \tfrac{1}{2} A_3 t^2 + A_2 t), \\
&\qquad\qquad\qquad (RA_2 + A_3 t, RA_3))
\end{aligned}$$

satisfies

$$j(A_4,((A_5,R),t))\cdot I_1((A_0,A_1),(A_2,A_3)) = I_1(\gamma((A_4,((A_5,R),t)((A_0,A_1),(A_2,A_3))$$

and gives rise to the canonical 2-co-chain $g_j{}^2$

$$g_j{}^2(((A_4{}^1,((A_5{}^1,R_1),t_1)),(A_4{}^2,((A_5{}^2,R_2),t_2))) = ((\tfrac{1}{120}A_5{}^1t_2{}^5+\tfrac{1}{24}A_4{}^1t_2{}^4,\tfrac{1}{24}A_5{}^1t_2{}^4+\tfrac{1}{6}A_4{}^1t_2{}^3), (\tfrac{1}{6}A_5{}^1t_2{}^3+\tfrac{1}{2}A_4{}^1t_2{}^2,\tfrac{1}{2}A_5{}^1t_2{}^2+A_4{}^1t_2))$$

Now, if 'J' embeds $\mathbf{R}^3\otimes\mathbf{R}_1{}^3$ into $(\mathbf{R}^3\otimes\mathbf{R}_1{}^3)\otimes(\mathbf{R}_2{}^3\otimes\mathbf{R}_3{}^3)$, then the function '$\gamma'$' defined by the rule $\gamma(A_4,((A_5,R),t))\circ J=J_0\gamma'(A_4,((A_5,R),t))$, is in $\mathrm{Hom}(A(W)_1{}^4,\mathrm{Aut}(\mathbf{R}^3\otimes\mathbf{R}_1{}^3))$. Which means that

$$(A(W)_5,\Phi\notin(\mathbf{R}^3\otimes\mathbf{R}^3)_0^{\mathrm{Ker}(\Phi)}$$

The function $g_j{}^3=\delta(g_j{}^2)$ is found to be

$$\begin{aligned}
&g_j{}^3((A_4{}^1,((A_5{}^1,R_1),t)),(A_4{}^2,((A_5{}^2,R_2),t_2)),(A_4{}^3,((A_5{}^3,R_3),t_3)))\\
&=((\tfrac{1}{24}(A_5{}^1t_2{}^4t_3+A_5{}^2t_3{}^4t_1)+(A_5\tfrac{1}{12}{}^1(t_2{}^3t_3{}^2+t_2{}^2t_3{}^3)+A_5{}^2(t_3{}^3t_1{}^2\\
&\quad+t_1{}^3t_3{}^2,+\tfrac{1}{6}(A_4{}^1(t_2{}^3t_3+t_2t_3)+A_4{}^2(t_3{}^3t_1+t_1{}^3t_3))\\
&\quad+\tfrac{1}{4}(A_4{}^1\,t_2{}^2\,t_3{}^2+A_4{}^2\,t_3{}^2\,t_1{}^2)),(\tfrac{1}{6}(A_5{}^1\,t_2{}^3t_3+A_5{}^2\,t_3{}^3\,t_1)\\
&\quad+\tfrac{1}{4}(A_5{}^1t_2{}^2t_3{}^2+A_3{}^2t_4{}^2t_3{}^2)+((\tfrac{1}{2}(A_4{}^1(t_2{}^2t_3+t_3{}^2t_2)\\
&\quad+A_4{}^2(t_3{}^2t_1+t_1{}^2t_3)),(\tfrac{1}{2}A_5{}^1t_2{}^2t_3+A_5{}^2t_3{}^2t_1+A_4{}^1t_2t_3+A_4{}^2t_3t_1,0))
\end{aligned}$$

Thus $g_j{}^3\neq0$ and $g_j{}^3\notin C_\gamma{}^3(A(W)_1{}^4,\mathbf{R}^3\otimes\mathbf{R}_1{}^3)$. This means that we must have

$$(A(W)_5,\Phi)\notin\Theta_1^{A(W)_5},(\mathbf{R}^3\otimes\mathbf{R}_1{}^3)_1^{A(W)_5}$$

One must therefore proceed to compute $g_j{}^4=\delta(g_j{}^3)$ in order to further categorise the prolongation. So far, the tedious calculation has not been performed. It has been performed for the sub-loop $\tilde{A}(W)_5$ of $A(W)_5$ with trivial rotations. Here we find the interesting result that

$$g_j{}^4=0,\qquad g_j{}^3\notin C^3_{\gamma'}(\tilde{A}(W)_5,\mathbf{R}^3\otimes\mathbf{R}_1{}^3)$$

We surmise then that

$$(\tilde{A}(W)_5,\Phi)\in\Theta_1^{\mathrm{Ker}(\Phi)},\qquad\notin\Theta_1^{\mathrm{Ker}(\Phi)}\cap(\mathbf{R}^3\otimes\mathbf{R}_1{}^3)_1^{A(W)_5}$$

Due to the very quickly rising length of calculation with 'm', we terminate our considerations of the loops

$$\langle A(W)_m\rangle\, m\in Z^+$$

here, for $m=5$. We have, however, gone far enough to see the trend of our work. As 'm' increases; the associativity of the loops $A(W)_m$ decreases. Specifically, for $m=0$ and 1, the loops $A(W)_0$ and $A(W)_1$ are associative groups; $A(W)_1$ being the Galilei group. For $m=2$, we encounter the first non-associative group. It is a prolongation of the simplest kind

$$(L,\Phi)\in K_0^{\mathrm{Ker}(\Phi)}$$

For $m = 3$, we encounter a less associative loop in the prolongation class

$$(L, \Phi) \in (K_1^{\mathrm{Ker}(\Phi)} - (K_1^{\mathrm{Ker}(\Phi)} \cap \Theta_1{}^L)) \subset K_1^{\mathrm{Ker}(\Phi)}$$

Again for $m = 4$, we find that

$$(L, \Phi) \in K_1{}^L \cap \Theta_1^{\mathrm{Ker}(\Phi)}$$

and finally, for $m = 5$

$$(L, \Phi) \in (\Theta_1^{\mathrm{Ker}(\Phi)} - (K_1{}^L \cap \Theta_1^{\mathrm{Ker}(\Phi)})) \subset \Theta_1^{\mathrm{Ker}(\Phi)}$$

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